FACTORING OPERATORS THROUGH HILBERT SPACE

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ABSTRACT

We give an example of a pair of Banach spaces E and F so that neither E' nor F has cotype 2, but every bounded linear operator from E into F factors through Hilbert space.

Let E and F be Banach spaces. Denote by L(E,F) the space of all bounded linear operators from E into F. An operator $T \in L(E,F)$ is said to factor through a Hilbert space if there is a Hilbert space H and operators $R \in L(H,F)$ and $S \in L(E,H)$ such that T=RS. The space of all such operators is denoted by $\Gamma_2(E,F)$. A considerable amount is known concerning the coincidence of L(E,F) and $\Gamma_2(E,F)$. For instance, if either E or F has the GL property, and if both E' and F are cotype 2, then $L(E,F) = \Gamma_2(E,F)$. If we do not assume the GL property, then it is known that every approximable operator in L(E,F) factors through a Hilbert space if both E' and F are cotype 2. Of course, $L(E,F) = \Gamma_2(E,F)$ is also true if either E or F is Hilbertian. All of the results quoted above may be found in the lecture notes of Pisier [P], where the factorization question is thoroughly discussed. In Remark 4.10 in [P], the converse direction to the above results is considered. More precisely, assume that every approximable operator from E to F belongs to $\Gamma_2(E,F)$, is one of the following conditions necessarily true:

- (a) E or F is Hilbertian,
- (b) E' and F have cotype 2?

In this note, we show that the answer to the above question is negative. The

Received March 1, 1990

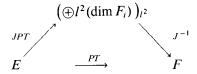
author thanks Professor Pisier for pointing out this question to him and for suggesting that he try the approach used below.

THEOREM 1. There are Banach spaces E and F with $L(E,F) = \Gamma_2(E,F)$ while neither E' nor F has cotype 2.

Of course, the cotype condition also rules out the possibility that either E or F is Hilbertian. The spaces E and F will be l^2 -sums of finite dimensional l^p spaces where p approaches 2. For $1 \le p \le \infty$, we denote by p' the conjugate: $p^{-1} + (p')^{-1} = 1$, $1' = \infty$, $\infty' = 1$. The crucial ingredient in the proof is supplied by the following result of Pełczyński and Rosenthal [PR]:

THEOREM 2. Let n be a natural number and let $\epsilon > 0$ be given. There is a $\delta(n, \epsilon) > 0$ such that if $|p-2| \le \delta(n, \epsilon)$, then every subspace G of l^p with dim $G \le n$ is $(1 + \epsilon)$ -complemented and $(1 + \epsilon)$ -isomorphic to $l^2(\dim G)$.

To prove Theorem 1, let $r(n) = 2 + \delta(n,1)$ for all $n \ge 1$. We define four sequences (p_i) , (q_i) , (m_i) and (n_i) inductively as follows. Let $q_1 = m_1 = n_1 = 1$, and let $p_1 = r(m_1)$. If p_i , q_i , m_i and n_i have been chosen for $i \le k$, let q_{k+1} be such that $q'_{k+1} = r(n_1 + \cdots + n_k)$. Now choose the integer m_{k+1} to satisfy $m_{k+1}^{1/q_{k+1}-1/2} \ge k+1$. Finally, let $p_{k+1} = r(m_1 + \cdots + m_{k+1})$ and then choose the integer n_{k+1} to satisfy $n_{k+1}^{1/2-1/p_{k+1}} \ge k+1$. Consider $E = (\bigoplus l^{q_i}(m_i))_{l^2}$ and $F = (\bigoplus l^{p_i}(n_i))_{l^2}$. It is clear that both E' and F fail cotype 2. Every $T \in L(E, F)$ has an obvious matrix representation $T = (T_{ij})$, where $T_{ij}: l^{q_j}(m_j) \to l^{p_i}(n_i)$. For all i, define $F_i = \text{range } T_{i1} + \cdots + \text{range } T_{ii} \subset l^{p_i}(n_i)$. Note that dim $F_i \le m_1 + \cdots + m_i$. By choice of p_i , F_i is 2-isomorphic to $l^2(\dim F_i)$ and 2-complemented in $l^{p_i}(n_i)$. Let $P_i: l^{p_i}(n_i) \to l^{p_i}(n_i)$ be a projection of norm ≤ 2 onto F_i , and let $J_i: F_i \to l^2(\dim F_i)$ be an isomorphism with $||J_i|| \le 1$, $||J_i^{-1}|| \le 2$. Then $P: F \to F$ given by $P(x_1, x_2, \dots) = (P_1 x_1, P_2 x_2, \dots)$ is a projection onto $(\bigoplus F_i)_{l^2}$ with norm ≤ 2 . Also $J: (\bigoplus F_i)_{l^2} \to (\bigoplus l^2(\dim F_i))_{l^2}$ given by $J(x_1, x_2, \dots) = (J_1 x_2, J_2 x_2, \dots)$ is an isomorphism with $||J|| \le 1$, $||J^{-1}|| \le 2$. Thus the diagram



commutes. Hence $PT \in \Gamma_2(E, F)$. Let (S_{ij}) be the matrix representation of the map T - PT, with $S_{ij} : l^{q_j}(m_j) \to l^{p_i}(n_i)$. A direct computation reveals that $S_{ij} = 0$ for $i \ge j$. Hence, using the choice of the sequence (q_i) , the above argument may

be applied to show that $(T-PT)' \in \Gamma_2(F',E')$. Consequently, $T-PT \in \Gamma_2(E,F)$. But then $T=(T-PT)+PT \in \Gamma_2(E,F)$. This completes the proof of Theorem 1.

REFERENCES

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