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FACTORING OPERATORS THROUGH HILBERT SPACE

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ABSTRACT
We give an example of a pair of Banach spaces E and F so that neither E' nor F has cotype 2, but every bounded linear operator from E into F factors through Hilbert space.

Let E and F be Banach spaces. Denote by $L(E,F)$ the space of all bounded linear operators from E into F. An operator $T \in L(E,F)$ is said to *factor* through a Hilbert space if there is a Hilbert space H and operators $R \in L(H,F)$ and $S \in L(E,H)$ such that $T = RS$. The space of all such operators is denoted by $\Gamma_2(E,F)$. A considerable amount is known concerning the coincidence of $L(E,F)$ and $\Gamma_2(E,F)$. For instance, if either E or F has the GL property, and if both E' and F are cotype 2, then $L(E,F) = \Gamma_2(E,F)$. If we do not assume the GL property, then it is known that every approximable operator in $L(E,F)$ factors through a Hilbert space if both E' and F are cotype 2. Of course, $L(E,F) = \Gamma_2(E,F)$ is also true if either E or F is Hilbertian. All of the results quoted above may be found in the lecture notes of Pisier [P], where the factorization question is thoroughly discussed. In Remark 4.10 in [P], the converse direction to the above results is considered. More precisely, assume that every approximable operator from E to F belongs to $\Gamma_2(E,F)$, is one of the following conditions necessarily true:

(a) E or F is Hilbertian,

(b) E' and F have cotype 2?

In this note, we show that the answer to the above question is negative. The

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THEOREM 1. *There are Banach spaces E and F with $L(E,F) = \Gamma_2(E,F)$ while neither E' nor F has cotype* 2.

Of course, the cotype condition also rules out the possibility that either E or F is Hilbertian. The spaces E and F will be l^2-sums of finite dimensional l^p spaces where p approaches 2. For $1 \le p \le \infty$, we denote by p' the conjugate: $p^{-1} + (p')^{-1} = 1$, $1' = \infty$, $\infty' = 1$. The crucial ingredient in the proof is supplied by the following result of Pełczyński and Rosenthal [PR]:

THEOREM 2. *Let n be a natural number and let $\epsilon > 0$ be given. There is a $\delta(n,\epsilon) > 0$ such that if $|p-2| \le \delta(n,\epsilon)$, then every subspace G of l^p with* $\dim G \le$ *n is $(1+\epsilon)$-complemented and $(1+\epsilon)$-isomorphic to* $l^2(\dim G)$.

To prove Theorem 1, let $r(n) = 2 + \delta(n,1)$ for all $n \ge 1$. We define four sequences (p_i), (q_i), (m_i) and (n_i) inductively as follows. Let $q_1 = m_1 = n_1 = 1$, and let $p_1 = r(m_1)$. If p_i, q_i, m_i and n_i have been chosen for $i \le k$, let q_{k+1} be such that $q'_{k+1} = r(n_1 + \cdots + n_k)$. Now choose the integer m_{k+1} to satisfy $m_{k+1}^{1/q_{k+1}-1/2} \ge k+1$. Finally, let $p_{k+1} = r(m_1 + \cdots + m_{k+1})$ and then choose the integer n_{k+1} to satisfy $n_{k+1}^{1/2-1/p_{k+1}} \ge k+1$. Consider $E = (\bigoplus l^{q_i}(m_i))_{l^2}$ and $F = (\bigoplus l^{p_i}(n_i))_{l^2}$. It is clear that both E' and F fail cotype 2. Every $T \in L(E,F)$ has an obvious matrix representation $T = (T_{ij})$, where $T_{ij}: l^{q_j}(m_j) \to l^{p_i}(n_i)$. For all i, define $F_i = \text{range}\, T_{i1} + \cdots + \text{range}\, T_{ii} \subset l^{p_i}(n_i)$. Note that $\dim F_i \le m_1 + \cdots + m_i$. By choice of p_i, F_i is 2-isomorphic to $l^2(\dim F_i)$ and 2-complemented in $l^{p_i}(n_i)$. Let $P_i: l^{p_i}(n_i) \to l^{p_i}(n_i)$ be a projection of norm ≤ 2 onto F_i, and let $J_i: F_i \to l^2(\dim F_i)$ be an isomorphism with $\|J_i\| \le 1$, $\|J_i^{-1}\| \le 2$. Then $P: F \to F$ given by $P(x_1, x_2, \ldots) = (P_1x_1, P_2x_2, \ldots)$ is a projection onto $(\bigoplus F_i)_{l^2}$ with norm ≤ 2. Also $J: (\bigoplus F_i)_{l^2} \to (\bigoplus l^2(\dim F_i))_{l^2}$ given by $J(x_1, x_2, \ldots) = (J_1x_2, J_2x_2, \ldots)$ is an isomorphism with $\|J\| \le 1$, $\|J^{-1}\| \le 2$. Thus the diagram

$$\begin{array}{ccc} & \left(\bigoplus l^2(\dim F_i)\right)_{l^2} & \\ {}^{JPT}\nearrow & & \searrow^{J^{-1}} \\ E & \xrightarrow{\quad PT \quad} & F \end{array}$$

commutes. Hence $PT \in \Gamma_2(E,F)$. Let (S_{ij}) be the matrix representation of the map $T - PT$, with $S_{ij}: l^{q_j}(m_j) \to l^{p_i}(n_i)$. A direct computation reveals that $S_{ij} = 0$ for $i \ge j$. Hence, using the choice of the sequence (q_j), the above argument may be applied to show that $(T - PT)' \in \Gamma_2(F',E')$. Consequently, $T - PT \in \Gamma_2(E,F)$. But then $T = (T - PT) + PT \in \Gamma_2(E,F)$. This completes the proof of Theorem 1.

References

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